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Time-Domain Finite Element Analysis of Viscoelastic Structures with Fractional Derivatives Constitutive Relations

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Numerical procedures for the time integration of the spatially discretized finite element equations for viscoelastic structures governed by a constitutive equation involving fractional calculus operators are presented. To avoid difficulties concerning fractional-order initial conditions, a form of the fractional calculus model of viscoelasticity involving a convolution integral with a singular memory kernel of Mittag–Leffler type is used. The constitutive equation is generalized to three-dimensional states for isotropic materials. A simplification of the fractional derivative of the memory kernel is used, in connection with Grünwald's definition of fractional differentiation and a backward Euler rule, for the time evolution of the convolution term. A desirable feature of this process is that no actual evaluation of the memory kernel is needed. This, together with the Newmark method for time integration, enables the direct calculation of the time evolution of the nodal degrees of freedom. To illustrate the ability of the numerical procedure a few numerical examples are presented. In one example the numerically obtained solution is compared with a time series expansion of the analytical solution.

Nomenclature

A	= cross-sectional area
$\boldsymbol{B}$	= strain-displacement matrix
b	= uniaxial relaxation constant
b_G	= shear relaxation constant
b_K	= bulk relaxation constant
D	= fractional differentiation operator
$\boldsymbol{d}(t)$	= nodal-degrees-of-freedom vector
E	= instantaneous uniaxial viscoelastic modulus
E_α	= α-order Mittag–Leffler function
E_∞	= relaxed uniaxial viscoelastic modulus
$e_{ij}(t)$	= deviatoric part of strain tensor
$\boldsymbol{F}(t)$	= external body force vector
$f_1(t)$	= shear memory kernel
$f_2(t)$	= bulk memory kernel
G	= instantaneous bulk modulus
$G_{\text{rel}}(t)$	= shear stress relaxation function
G_∞	= relaxed shear modulus
K_∞	= instantaneous bulk modulus
$\boldsymbol{K}$	= stiffness matrix
$K_{\text{rel}}(t)$	= bulk relaxation function
K_∞	= relaxed bulk modulus
k_∞	= instantaneous spring constant
k_∞	= relaxed spring constant
L	= length
$\boldsymbol{M}$	= mass matrix
m	= mass
$\boldsymbol{N}$	= shape function matrix
$\boldsymbol{R}(t)$	= external load vector
s	= Laplace domain variable
$s_{ij}(t)$	= deviatoric part of stress tensor
t	= time
$\boldsymbol{u}(t)$	= displacement vector
u_0	= displacement amplitude
V	= volume
v_0	= velocity amplitude
α	= uniaxial fractional derivative exponent
α_G	= fractional derivative exponent in shear
α_K	= fractional derivative exponent in compression
β, γ	= parameters in the Newmark method
Γ	= gamma function
ΔE	= uniaxial viscoelastic relaxation strength
ΔG	= shear relaxation strength
ΔK	= bulk relaxation strength
$\Delta \boldsymbol{K}$	= relaxation strength matrix
Δk	= spring relaxation strength
Δt	= time step
δ_{ij}	= Kronecker delta
$\varepsilon_{ij}(t)$	= strain tensor
ν	= Poisson's ratio
ρ	= density
$\sigma_{ij}(t)$	= stress tensor
$\boldsymbol{\Phi}(t)$	= external surface traction vector
ω	= angular frequency

Subscript

k	= element number

Superscript

e	= elemental

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I. Introduction

CLASSICAL linear viscoelasticity can be formulated in three different forms. The first form involves integer derivative operators acting on both stress and strain.[1] To describe the dynamic properties of materials with weak frequency dependence over a broad frequency range requires a great number of high-order time derivatives acting on both stress and strain. This makes the model cumbersome to use and leads to a high-order equation of motion when incorporated into structural dynamics.

The second form involves convolution integrals.[1] To match the integer derivative model, the kernels should be taken as exponentially decaying terms. Once more, to describe weak frequency dependence, many terms are required, which makes the model cumbersome.

The third form uses internal variables and has been introduced into structural dynamics by Lesieutre and Mingori[2] and more recently generalized to three-dimensional states by Lesieutre and Bianchini.[3] The constitutive relation is then formulated as a set of coupled equations. To match with the integer derivative form, the evolution equations for the internal variables should be formulated as first-order differential equations in time. A similar viscoelastic formulation, used in finite element modeling, is the Golla–Hughes–McTavish minioscillator model.[4,5] The dissipative behavior is represented by

transform-based dissipation coordinates at element level. Again, many internal variables and parameters are required to match weak frequency dependence, which can make these models also cumbersome to use.

The constitutive modeling can also be done in the frequency domain by simply curve fitting a complex modulus to steady-state experimental data in a limited frequency band. There exist several conditions on the complex modulus that have to be satisfied for the model to have a reasonable representation in the time domain.[6] This makes the complex modulus model unsuitable to use in time-domain calculations.

By introducing fractional derivative operators instead of integer derivative operators in the constitutive equation of viscoelasticity, the number of terms and parameters required to model weak frequency dependence can be reduced significantly. The simplest model, able to describe finite initial stiffness, contains only a single fractional derivative operator acting on both stress and strain and only four material parameters. This model is commonly referred to as the fractional calculus model of viscoelasticity, and it has successfully been fitted to experimental data over broad frequency bands for several polymers.[7]

When the fractional calculus model of viscoelasticity, as formulated with fractional derivative operators, is incorporated directly into the framework of structural dynamics, the resulting time-differential equation of motion is of a fractional order higher than two. Hence, the fractional higher-order equation of motions requires initial conditions on nonphysical quantities of fractional order (e.g., Refs. 6 and 8). An additional form of the fractional calculus model of viscoelasticity has recently been introduced into structural dynamics to avoid the difficulties concerning the physical interpretation, justification, and verification of fractional-order initial conditions. This form uses a convolution integral formulation and singular memory kernel of Mittag–Leffler function type.[9]

Bagley and Torvik[7] developed a Laplace-domain finite element formulation for fractional calculus viscoelasticity using the derivative operator approach. This formulation is used by Bagley and Torvik[10] when studying transient response of a damped beam. Inverse Laplace transforms must then be numerically calculated at every point in time for which the response is demanded. Bagley and Calico[11] and Fenander[12] expanded the finite element (FE) equations of motion by introducing a fractional-order state-space vector. The resulting equations were decoupled by the complex eigenvectors of the system. Time-domain responses were then obtained by a modal synthesis approach. The major drawbacks with this formulation is that the size of the problem is heavily expanded, and only small one-dimensional problems have been studied.

The present study considers the development and solution of the spatially discretized system of dynamic FE equations for a viscoelastic structure having a constitutive equation of fractional calculus type. The fractional calculus model of viscoelasticity in the convolution integral form is generalized to three dimensions and used when formulating the FE equations. The resulting system of FE equations is a system of integro-differential equations of second order in time. An expression for the fractional derivative of the convolution term derived by Enelund and Olsson[6] in connection with the Grünwald definition of fractional differentiation is used for time evolution of the convolution term in the FE equations. A very desirable feature of this procedure is that no actual evaluation of the memory kernel is needed. Combining this with Newmark's method for time integration of the nodal degrees of freedom enables us to directly calculate the unknown displacements. The present formulation leads to a somewhat more convenient numerical procedure for the time integration than the numerical procedure developed by Padovan,[13] which employs the fractional derivative operator approach.

A few numerical examples are given to demonstrate the appearance of the algorithm. In one case the numerically obtained solution is compared to short time series expansions of analytically obtained solution where the order of the fractional differentiation in the constitutive relation is equal to a half.

II. Fractional Calculus Model of Viscoelasticity

For a viscoelastic material, the stress is not only a function of the actual strain but also of the previous strain history. Isothermal conditions and isotropic materials are assumed throughout the study. Cartesian components of tensors are denoted by subscript indices, and the summation convention for tensors is used. It is then convenient to introduce the deviatoric parts, s_{ij} of the stress tensor σ_{ij} and e_{ij} of the strain tensor ε_{ij}, as

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}, \qquad s_{ii} = s_{11} + s_{22} + s_{33} = 0 \tag{1a}$$

$$e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\delta_{ij}\varepsilon_{kk}, \qquad e_{ii} = e_{11} + e_{22} + e_{33} = 0 \tag{1b}$$

where $\frac{1}{3}\delta_{ij}\sigma_{kk}$ and $\frac{1}{3}\delta_{ij}\varepsilon_{kk}$ are the hydrostatic parts of the stress tensor and the strain tensor, respectively. Three-dimensional constitutive equations for an isotropic linear viscoelastic material following this decomposition can be written as two uncoupled sets of equations. A three-dimensional generalization of the fractional calculus model of viscoelasticity in fractional derivative operator form, according to Bagley and Torvik,[7] is

$$s_{ij}(t) + b_G^{\alpha_G} D^{\alpha_G} s_{ij}(t) = 2G_\infty e_{ij}(t) + 2G\, b_G^{\alpha_G} D^{\alpha_G} e_{ij}(t) \qquad 0 < \alpha_G < 1 \tag{2a}$$

$$\sigma_{kk}(t) + b_K^{\alpha_K} D^{\alpha_K} \sigma_{kk}(t) = 3K_\infty \varepsilon_{kk}(t) + 3K b_K^{\alpha_K} D^{\alpha_K} \varepsilon_{kk}(t) \qquad 0 < \alpha_K < 1 \tag{2b}$$

Here G and G_∞ are identified as the direct shear modulus and long-time shear modulus, respectively $\{G = E/[2(1+\nu)]\}$, whereas K and K_∞ are identified as the instantaneous bulk modulus and long-time bulk modulus, respectively, and $K = E/[3(1-2\nu)]$. Moreover, D^α is a generalized differentiation operator of order α, and b_G and α_G are the relaxation constant and fractional order of differentiation in shear, respectively, whereas b_K and α_K are the corresponding relaxation constant and fractional order of differentiation in compression, respectively. Note that, by letting $\alpha_K = \alpha_G = 1$ in Eqs. (2a) and (2b), the three-dimensional isotropic generalization of the three-parameter model of linear viscoelasticity is retained.

A suitable definition of differentiation of general order for the application to viscoelasticity is[14]

$$D^\alpha x(t) \equiv \frac{\mathrm{d}^N}{\mathrm{d}t^N}\left[\frac{1}{\Gamma(\rho)}\int_0^t (t-\tau)^{(\rho-1)} x(\tau)\,\mathrm{d}\tau\right], \qquad \alpha = N - \rho \tag{3}$$

where $0 < \rho \leq 1$, N is an integer that satisfies $\alpha < N \leq \alpha + 1$, and

$$\Gamma(x) \equiv \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}t, \qquad \Re(x) > 0 \tag{4}$$

where $\Re$ denotes the real part. Making the definition in Eq. (3) more specialized to $\alpha \in (0, 1)$ (the interval of interest for application to viscoelasticity) yields

$$D^\alpha x(t) \equiv \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t \frac{x(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau\right] \tag{5}$$

Because Eqs. (2a) and (2b) are differential equations (with the present definition of fractional-order differentiation), they naturally require initial conditions. The formal initial conditions are found to be[9]

$$D^{-(1-\alpha_G)} s_{ij}(0^+) = 2G D^{-(1-\alpha_G)} e_{ij}(0^+) \tag{6a}$$

$$D^{-(1-\alpha_K)} \sigma_{kk}(0^+) = 3K D^{-(1-\alpha_K)} \varepsilon_{kk}(0^+) \tag{6b}$$

These initial conditions are for $\alpha_G, \alpha_K \in (0, 1)$, a fractional integrated form of the three-dimensional isotropic Hooke's elastic law and, thus, weaker conditions than Hooke's law. Note that the given initial conditions allow for singularities in the strain histories. However, the work done to deform the material should be bounded for states where the linear viscoelastic model applies, which disqualifies unbounded strains, i.e.,

$$\left|\int_0^t \sigma_{ij}(\hat{t})\dot{\varepsilon}_{ij}(\hat{t})\,\mathrm{d}\hat{t}\right| < \infty \rightarrow |\varepsilon_{ij}(0^+)| < \infty \tag{7}$$

Keeping this in mind, the initial condition to Eqs. (2a) and (2b) can be taken as

$$s_{ij}(0^+) = 2Ge_{ij}(0^+) \tag{8a}$$

$$\sigma_{kk}(0^+) = 3K\varepsilon_{kk}(0^+) \tag{8b}$$

which are the elastic Hooke's law. Equations (8a) and (8b) trivially imply the initial conditions in Eqs. (6a) and (6b) because these conditions are integral conditions.

The equivalence to Eqs. (2a) and (2b) with initial condition equations (6a) and (6b) in convolution integral form is obtained by applying a Laplace transform to Eqs. (2a) and (2b) and subsequent inverse transform as

$$s_{ij}(t) = 2Ge_{ij}(t) - 2\Delta G\int_0^t f_1(t-\tau)e_{ij}(\tau)\,\mathrm{d}\tau \tag{9a}$$

where

$$f_1(t) = \frac{1}{b_G}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(t/b_G)^{\alpha_G n-1}}{\Gamma(\alpha_G n)}, \qquad t>0 \tag{9b}$$

and

$$\sigma_{kk}(t) = 3K\varepsilon_{kk}(t) - 3\Delta K\int_0^t f_2(t-\tau)\varepsilon_{kk}(\tau)\,\mathrm{d}\tau \tag{10a}$$

where

$$f_2(t) = \frac{1}{b_K}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(t/b_K)^{\alpha_K n-1}}{\Gamma(\alpha_K n)}, \qquad t>0 \tag{10b}$$

where $\Delta K = K - K_\infty$ is the bulk relaxation strength and $\Delta G = G - G_\infty$ is the shear relaxation strength. For α_G and $\alpha_K \in (0, 1)$, the memory kernels $f_1(t)$ and $f_2(t)$ are singular but the singularities are weak, and they are integrable. For a full derivation of the correspondence to Eqs. (9b) and (10b) in the one-dimensional case and a discussion of the properties, e.g., the regularity and the convergence, of the memory kernels, see Ref. 6. Note that initial conditions are automatically accounted for in Eqs. (9a) and (10a). Further, because the strains are bounded [see Eq. (7)], we have $(f_1 * e_{ij})(0^+) = (f_2 * \varepsilon_{kk})(0^+) = 0$, and the model predicts an instantaneous constitutive response following Hooke's law.

The infinite series in the memory kernels can be expressed in Mittag–Leffler notation, and the kernels can be written as

$$f_1(t) = -\frac{\mathrm{d}E_{\alpha_G}[-(t/b_G)^{\alpha_G}]}{\mathrm{d}t}, \qquad t>0 \tag{11a}$$

and

$$f_2(t) = -\frac{\mathrm{d}E_{\alpha_K}[-(t/b_K)^{\alpha_K}]}{\mathrm{d}t}, \qquad t>0 \tag{11b}$$

where E_α is the α-order Mittag–Leffler function[15]

$$E_\alpha[x] = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(1+\alpha k)} \tag{12}$$

Specializing to $\alpha_G = \alpha_K = 1$ gives

$$f_1(t) = \frac{1}{b_G}e^{-(t/b_G)}, \qquad f_2(t) = \frac{1}{b_K}e^{-(t/b_K)}, \qquad t>0 \tag{13}$$

which are the exponentially decaying memory kernels corresponding to the three-dimensional generalization of the three-parameter model of linear viscoelasticity, as expected. Using integration by parts, we rewrite Eqs. (9) and (10) as

$$s_{ij}(t) = \int_{0-}^{t} G_{\mathrm{rel}}(t-\tau)\dot{e}_{ij}(\tau)\,\mathrm{d}\tau \tag{14a}$$

where

$$G_{\mathrm{rel}}(t) = 2G_\infty + 2\Delta G E_{\alpha_G}[-(t/b_G)^{\alpha_G}], \qquad \geq 0 \tag{14b}$$

and

$$\sigma_{kk}(t) = \int_{0-}^{t} K_{\mathrm{rel}}(t-\tau)\dot{\varepsilon}_{kk}(\tau)\,\mathrm{d}\tau \tag{15a}$$

where

$$K_{\mathrm{rel}}(t) = 3K_\infty + 3\Delta K E_{\alpha_K}[-(t/b_K)^{\alpha_K}], \qquad t \geq 0 \tag{15b}$$

where a superposed dot denotes one integer differentiation with respect to time, whereas $G_{\mathrm{rel}}(t)$ and $K_{\mathrm{rel}}(t)$ are the stress relaxation functions in deviatoric and volumetric deformations, i.e., the stress responses on a unit step deviatoric strain and a unit step volumetric strain imposed at time $t = 0$, respectively. Singular kernels of Mittag–Leffler function type were introduced into viscoelasticity by Rabotnov.[16] Koeller[17] established this kind of kernel relationship to the fractional derivative model of viscoelasticity.

Bagley and Torvik[18] showed that the uniaxial fractional calculus model of viscoelasticity predict a nonnegative rate of energy dissipation and nonnegative internal work during harmonic excitations if the parameters are subjected to some restrictions. For the present three-dimensional isotropic formulation, the equivalence to the Bagley and Torvik[18] restrictions [for α_G and $\alpha_K \in (0, 1)$] are

$$K > K_\infty > 0, \qquad G > G_\infty > 0, \qquad b_G > 0, \qquad b_K > 0 \tag{16}$$

Further, in the case of arbitrary loading, Rabotnov[16] demanded that the uniaxial relaxation function should be a positive kernel, i.e., if (in the isotropic case) $G_{\mathrm{rel}}(t)$, $K_{\mathrm{rel}}(t) \geq 0$, and $t \geq 0$; $G_{\mathrm{rel}}(t)$, $K_{\mathrm{rel}}(t)$ are monotonic decreasing functions tending to finite limits as $t \to \infty$ and $G_{\mathrm{rel}}(t)$, $K_{\mathrm{rel}}(t)$ are convex downward for the dissipation or internal work to be nonnegative. These requirements are satisfied by the relaxation functions [Eqs. (14b) and (15b)] corresponding to the fractional calculus model for α_G and $\alpha_K \in (0, 1)$ if the parameters are subjected to the restrictions in Eq. (16). The reasons for this are that the Mittag–Leffler function $E_\alpha[-x]$ is a completely monotonic function for $0 < \alpha \leq 1$ and $x \geq 0$, i.e. (Ref. 15),

$$(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}(E_\alpha[-x]) \geq 0, \qquad n = 1, 2, 3, \ldots \tag{17}$$

and

$$\lim_{x\to 0} E_\alpha[x] = 1, \qquad \lim_{x\to\infty} E_\alpha[x] = 0 \tag{18}$$

Moreover, from Eq. (17) it also follows that the fractional calculus model represents a fading memory in a rather strict sense because the memory kernels [Eqs. (9b) and (10b)] are monotonically decreasing functions of time, i.e., $\mathrm{d}f_1(t)/\mathrm{d}t$ and $\mathrm{d}f_2(t)/\mathrm{d}t < 0$.

The following interesting properties of the convolution integrals in Eqs. (9b) and (10b) are useful for the numerical evaluation of the convolution (for a derivation, see Ref. 6):

$$D^{\alpha_G}(f_1 * e_{ij})(t) = -(1/b_G^{\alpha_G})[(f_1 * e_{ij})(t) - e_{ij}(t)] \tag{19a}$$

$$D^{\alpha_K}(f_2 * \varepsilon_{kk})(t) = -(1/b_K^{\alpha_K})[(f_2 * \varepsilon_{kk})(t) - \varepsilon_{kk}(t)] \tag{19b}$$

III. Structural Analysis

Now we restrict ourselves to polycrystalline materials with random orientated fibers for which the order of fractional differentiations and the relaxation constants in deviatoric and hydrostatic relaxation may be assumed to be equal, i.e., $\alpha_G = \alpha_K = \alpha$ and $b_G = b_K = b$. The constitutive relation equations (9a) and (10a) can then be written as a single equation (in matrix notation):

$$\sigma(t) = K\boldsymbol{M}_K[\varepsilon(t) - (\Delta K/K)(f * \varepsilon)(t)] + G\boldsymbol{M}_G \times [\varepsilon(t) - (\Delta G/G)(f * \varepsilon)(t)] \tag{20}$$

where

$$\boldsymbol{M}_K = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{M}_G = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and components σ_{ij} and ε_{ij} are collected in column vectors σ and ε where strains and stresses are expressed using compressed matrix notation following the ordering convention for strains,

$$\varepsilon_1 = \varepsilon_{11}, \qquad \varepsilon_2 = \varepsilon_{22}, \qquad \varepsilon_3 = \varepsilon_{33}$$

$$\varepsilon_4 = 2\varepsilon_{23}, \qquad \varepsilon_5 = 2\varepsilon_{31}, \qquad \varepsilon_6 = 2\varepsilon_{12}$$

and for stresses,

$$\sigma_1 = \sigma_{11}, \qquad \sigma_2 = \sigma_{22}, \qquad \sigma_3 = \sigma_{33}$$

$$\sigma_4 = \sigma_{23}, \qquad \sigma_5 = \sigma_{31}, \qquad \sigma_6 = \sigma_{12}$$

Moreover, in this case Eqs. (19a) and (19b) can be written (in matrix notation) as

$$D^\alpha(f * \varepsilon)(t) = -(1/b^\alpha)[(f * \varepsilon)(t) - \varepsilon(t)] \tag{21}$$

with

$$(f * \varepsilon)(0^+) = \mathbf{0}$$

where

$$f(t) = \frac{1}{b}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(t/b)^{\alpha n-1}}{\Gamma(\alpha n)}, \qquad t > 0 \tag{22}$$

We will now discretize the structure studied by employing a standard displacement-based FE formulation. The assumed displacement field within an element is expressed in terms of interpolations of corresponding element nodal displacements as

$$\boldsymbol{u}^e(x, y, z, t) = \boldsymbol{N}^e(x, y, z)\boldsymbol{d}^e(t) \tag{23}$$

where $(\boldsymbol{u}^e)^T = (u_1, u_2, u_3)$ is the displacement vector and $(\ )^T$ denotes matrix transposition, $\boldsymbol{d}^e$ is the element nodal degrees of freedom vector, and $\boldsymbol{N}^e$ is the element shape function matrix containing polynominal shape functions. The strains are related to the element nodal displacements as

$$\varepsilon(x, y, z, t) = \partial\,\boldsymbol{u}^e(x, y, z, t) = \partial\,\boldsymbol{N}^e(x, y, z)\boldsymbol{d}^e(t)$$

$$= \boldsymbol{B}^e(x, y, z)\boldsymbol{d}^e(t) \tag{24}$$

where $\boldsymbol{B}^e$ is the element strain-displacement matrix and ∂ is the appropriate spatial derivative matrix operator (e.g., Ref. 19).

Equations governing the dynamic response of a viscoelastic structure can be derived by using the principle of virtual work, which states that for any compatible displacements the total internal work is equal to the total work done by external loads. Using the FE formulations for the element displacements and strains in Eqs. (23) and (24) and the constitutive relation in Eq. (20), together with the principle of virtual work, yields the spatially discretized system of FE equations

$$\boldsymbol{M}\ddot{\boldsymbol{d}}(t) + \boldsymbol{K}\boldsymbol{d}(t) - \Delta\boldsymbol{K}(f * \boldsymbol{d})(t) = \boldsymbol{R}(t) \tag{25}$$

where $\boldsymbol{M}$ is the consistent mass matrix, so that with ρ_k the density of element k and the number of elements (NEL),

$$\boldsymbol{M} = \sum_{k=1}^{\mathrm{NEL}}\int_{V_k^e}(\boldsymbol{N}_k^e)^T\rho_k\boldsymbol{N}_k^e\,\mathrm{d}V \tag{26}$$

$$\boldsymbol{K} = \sum_{k=1}^{\mathrm{NEL}}\int_{V_k^e}(\boldsymbol{B}_k^e)^T(K_k\boldsymbol{M}_K + G_k\boldsymbol{M}_G)\boldsymbol{B}_k^e\,\mathrm{d}V \tag{27}$$

$$\Delta\boldsymbol{K} = \sum_{k=1}^{\mathrm{NEL}}\int_{V_k^e}(\boldsymbol{B}_k^e)^T(\Delta K_k\boldsymbol{M}_K + \Delta G_k\boldsymbol{M}_G)\boldsymbol{B}_k^e\,\mathrm{d}V \tag{28}$$

consistent with external body forces $\boldsymbol{F}$ and surface tractions $\boldsymbol{\Phi}$,

$$\boldsymbol{R}(t) = \sum_{k=1}^{\mathrm{NEL}}\int_{V_k^e}(\boldsymbol{N}_k^e)^T\boldsymbol{F}_k^e(t)\,\mathrm{d}V + \sum_{k=1}^{\mathrm{NEL}}\int_{S_k^e}(\boldsymbol{N}_k^e)^{\mathrm{T}}\boldsymbol{\Phi}_k^e(t)\,\mathrm{d}S \tag{29}$$

Note that the preceding system of FE equations is a system of second-order integro-differential equations in time and, thus, only requires initial conditions on the physical quantities $\boldsymbol{d}$ and $\dot{\boldsymbol{d}}$, whereas the inclusion of the fractional derivative form in Eqs. (2a) and (2b) into an FE formulation leads to a system of differential equations of order $2 + \alpha$ in time and requires initial conditions on $D^{\alpha-1}\boldsymbol{d}$, $D^\alpha\boldsymbol{d}$, and $D^{\alpha+1}\boldsymbol{d}$ besides conditions on the physical quantities. Another advantage of the present formulation is that structures that are viscoelastic in some parts only can be easily handled and the convolution term needs only to be calculated for nodal displacements in viscoelastic parts.

With a suitable discretization of the time derivatives in Eq. (25), the equations of motion can be solved for discrete times. Note, though, that special consideration must be given to the convolution term because it, formally, contains information of the entire time history of the strain. Hence, in parallel with solving the discretized system of FE equations [Eq. (25)], one has to find the time evolution of the convolution term. Introducing the assumed strain field equation (24) into Eq. (21) and noting that the convolution is a scalar operation gives

$$\boldsymbol{B}^e\{D^\alpha(f * \boldsymbol{d}^e)(t) + (1/b^\alpha)[(f * \boldsymbol{d}^e)(t) - \boldsymbol{d}^e(t)]\} = \mathbf{0} \tag{30}$$

Now, multiplying Eq. (30) from the left with the matrix transpose $(\boldsymbol{B}^e)^T$ from the left, integrating over the element volume, and summarizing contributions from all of the elements gives

$$\boldsymbol{C}\{D^\alpha(f * \boldsymbol{d})(t) + (1/b^\alpha)[(f * \boldsymbol{d})(t) - \boldsymbol{d}(t)]\} = \mathbf{0} \tag{31}$$

where $\boldsymbol{C}$ is a quadratic matrix

$$\boldsymbol{C} = \sum_{k=1}^{\mathrm{NEL}}\int_{V_k^e}(\boldsymbol{B}_k^e)^T\boldsymbol{B}_k^e\,\mathrm{d}V \tag{32}$$

which means that a possible FE approximation of Eq. (21) will be

$$D^\alpha(f * \boldsymbol{d})(t) = -(1/b^\alpha)[(f * \boldsymbol{d})(t) - \boldsymbol{d}(t)] \tag{33}$$

The first term in Eq. (33) is a column vector containing convolutions of the active nodal degrees of freedom $\boldsymbol{d}$ for the structure. In practice, the structure will be restrained so that the quadratic matrix $\boldsymbol{C}$ will never be singular. By multiplying Eq. (30) with $(\boldsymbol{B}^e)$ we employ, in principle, a Petrov–Galerkin method for determining an FE formulation of the time evolution of the convolution term. We might also consider Eq. (33) as the convolution evolution equation fulfilled in a least square sense.

Note also that the assumptions of equal fractional order and relaxation constants considerably simplify the FE discretization of the evolution of the convolution term.

IV. Time-Integration Algorithm

Consider, first, the discretized time evolution of the convolution term. A possible approximation of the fractional-order time derivative of a function $x(t)$ is obtained by a truncation of Grünwald's definition of fractional differentiation[14]:

$$(D^\alpha x)_{n+1} = \frac{1}{(\Delta t)^\alpha}\sum_{j=0}^{n}B_j(\alpha)x_{n+1-j} \tag{34}$$

with

$$B_j(\alpha) = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}$$

It is assumed that the spacing in time is uniform, i.e., $x(t) = x(n\Delta t)$. The calculations of $B_j(\alpha)$ are simplified by the recursion formula

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} = \frac{(j-1-\alpha)}{j}\,\frac{\Gamma(j-1-\alpha)}{\Gamma(j)} \tag{35}$$

For convenience we rewrite the expression in Eq. (34) as

$$(D^{\alpha}x)_{n+1} = \frac{1}{(\Delta t)^{\alpha}}\left[x_{n+1} + \sum_{j=1}^{n} B_j(\alpha)\, x_{n+1-j} \right] \tag{36}$$

With the application of the backward Euler approximation (in time) to the relation for the convolution term in Eq. (33) while using Eq. (36), we obtain a possible approximation for the convolution term in Eq. (25) as

$$(f * \boldsymbol{d})_{n+1} = \frac{(\Delta t)^{\alpha}}{(\Delta t)^{\alpha} + b^{\alpha}} \times \left[\boldsymbol{d}_{n+1} - \left(\frac{b}{\Delta t} \right)^{\alpha} \sum_{j=1}^{n} B_j(\alpha)\, (f * \boldsymbol{d})_{n+1-j} \right] \tag{37}$$

and the initial value $(f * \boldsymbol{d})(0^+) = \boldsymbol{0}$. Note that the whole time history of the convolution term $(f * \boldsymbol{d})$ for each nodal degree of freedom must formally be saved and included in each time step and that the nodal displacements should be known quantities at time t_{n+1}. Note that no actual calculation of the convolution term or evaluation of the memory kernel is needed; this is a very desirable feature because Mittag–Leffler functions do not converge easily.

Using the Newmark method for time integration, the nodal displacement solution at time t_{n+1} is obtained from[19]

$$\boldsymbol{d}_{n+1} = \boldsymbol{d}_n + \Delta t\, \dot{\boldsymbol{d}}_n + \frac{(\Delta t)^2}{2}[(1-2\beta)\, \ddot{\boldsymbol{d}}_n + 2\beta\, \ddot{\boldsymbol{d}}_{n+1}] \tag{38a}$$

and the nodal velocities from

$$\dot{\boldsymbol{d}}_{n+1} = \dot{\boldsymbol{d}}_n + \Delta t[(1-\gamma)\, \ddot{\boldsymbol{d}}_n + \gamma\, \ddot{\boldsymbol{d}}_{n+1}] \tag{38b}$$

with initial conditions on $\boldsymbol{d}_0$ and $\dot{\boldsymbol{d}}_0$. The parameters β and γ are chosen to control stability and accuracy. Using the preceding approximations [Eqs. (38a) and (38b)], together with the approximation for the convolution term in Eq. (37) and the system of FE equations [Eq. (25)] evaluated at the end of each time step (that is, an implicit approach),

$$\boldsymbol{M}\ddot{\boldsymbol{d}}_{n+1} + \boldsymbol{K}\boldsymbol{d}_{n+1} - \Delta\boldsymbol{K}(f * \boldsymbol{d})_{n+1} = \boldsymbol{R}_{n+1} \tag{39}$$

makes it possible to calculate the unknown nodal displacements and their time derivatives.

The initial acceleration $\ddot{\boldsymbol{d}}_0$ is first calculated from Eq. (39). By combining Eqs. (38a), (39), and (37) the nodal displacement solution $\boldsymbol{d}_{n+1}$ is obtained from

$$\boldsymbol{d}_{n+1} = \boldsymbol{A}\left[\boldsymbol{d}_n + \Delta t\, \dot{\boldsymbol{d}}_n + \frac{(\Delta t)^2}{2}(1-2\beta)\, \ddot{\boldsymbol{d}}_n + (\Delta t)^2 \beta \boldsymbol{M}^{-1}\, \boldsymbol{R}_{n+1} - \frac{(\Delta t)^2 \beta b^{\alpha}}{(\Delta t)^{\alpha} + b^{\alpha}} \boldsymbol{M}^{-1} \Delta\boldsymbol{K} \sum_{j=1}^{n} B_j(\alpha)\, (f * \boldsymbol{d})_{n+1-j} \right] \tag{40}$$

with

$$\boldsymbol{A} = \left\{ \boldsymbol{I} + (\Delta t)^2 \beta (\boldsymbol{M})^{-1} \left[\boldsymbol{K} - \frac{(\Delta t)^{\alpha}}{(\Delta t)^{\alpha} + b^{\alpha}} \Delta\boldsymbol{K} \right] \right\}^{-1}$$

where $\boldsymbol{I}$ is the identity matrix. Using Eq. (37), we obtain $(f * \boldsymbol{d})_{n+1}$. Inserting this into Eq. (39) gives $\ddot{\boldsymbol{d}}_{n+1}$. Using Eq. (38b), we obtain $\dot{\boldsymbol{d}}_{n+1}$, and a new time step can be taken.

Letting $\beta = 0$ and $\gamma = \frac{1}{2}$ (which implies no algorithmic damping) in Newmark's method [Eqs. (38a) and (38b)] gives a nearly explicit method, and the procedure for calculating the unknown nodal degrees of freedom becomes very simple. Again, calculate the initial acceleration $\ddot{\boldsymbol{d}}_0$ from Eq. (39), and $\boldsymbol{d}_{n+1}$ is then obtained directly from Eq. (38a). The insertion of $\dot{\boldsymbol{d}}_{n+1}$ into Eq. (37) gives $(f * \boldsymbol{d})_{n+1}$, and the updated acceleration $\ddot{\boldsymbol{d}}_{n+1}$ is now obtained from Eq. (39). Using this, we obtain the updated velocity $\dot{\boldsymbol{d}}_{n+1}$ from Eq. (38b), and a new time step can be taken. The nearly explicit method is only conditionally stable. For an elastic material, i.e., in this case letting $\Delta G = \Delta K = 0$, the critical time step is

$$\Delta t_{\text{crit}}^{\text{elastic}} = 2/\omega_{\max} \tag{41}$$

Here $\omega_{\max}$ is the highest elastic or undamped eigenfrequency of the spatially discretized system [found by letting $\Delta G = \Delta K = 0$ in Eq. (25) and solving the corresponding eigenvalue problem]. This is believed to be conservative for the viscoelastic structure because the viscoelastic material weakens with increasing time, which results in decreasing $\omega_{\max}$.

V. Numerical Examples

Numerical examples are presented to show the purpose and verify the accuracy and convergence of the present numerical procedure.

A. Single-Degree-of-Freedom System

A one-degree-of-freedom system consisting of a discrete mass and a viscoelastic spring governed by fractional derivative law, formulated in analogy with Eq. (20), is considered. Similar examples have also been studied by Enelund and Olsson.[9] The equation of motion for the system is

$$m\ddot{u}(t) + ku(t) - \Delta k(f * u) = R(t) \tag{42}$$

with

$$u(0^+) = u_0 \qquad \text{and} \qquad \dot{u}(0^+) = v_0$$

where $R(t)$ is the applied force and $u(t)$ the displacement, whereas $f(t)$ is the memory kernel in Eq. (22).

Consider the case of $\alpha = \frac{1}{2}$ and an impulsive load applied at time $t = 0$. As far as Eq. (42) is concerned, this is equivalent to having zero load but specifying a nonvanishing initial velocity and zero initial displacement. That is,

$$\begin{aligned} R(t) &= 0, \qquad t > 0 \\ u(0^+) &= 0, \qquad \dot{u}(0^+) = v_0 \end{aligned} \tag{43}$$

Note that, though the constitutive model employed requires knowledge of the previous history to predict the actual state of stress, nonvanishing initial conditions can be prescribed without presupposing the previous displacement history. The reason for this is that any initial values of the displacement and the velocity can be accomplished by applying sufficiently large forces in an arbitrarily short time interval preceding $t = 0$. The system is supposed to be load free a long time before applying any load.

A straightforward application of the Laplace transform to Eq. (42), with load and initial conditions according to Eq. (43), yields the Laplace-domain displacement solution

$$u(s) = mv_0\left[ms^2 + \frac{k - \Delta k + k(bs)^{\frac{1}{2}}}{1 + (bs)^{\frac{1}{2}}} \right]^{-1} \tag{44}$$

An asymptotic series as $s \to \infty$ is

$$u(s) \sim \frac{v_0}{s^2} - \frac{v_0 k}{ms^4} + \frac{v_0(k - k_\infty)}{mb^{\frac{1}{2}} s^{\frac{9}{2}}} + \mathcal{O}\left(\frac{1}{s^5} \right) \tag{45}$$

A termwise Laplace inversion yields the first few terms in a series expansion of $u(t)$ as

$$u(t) = v_0 t - \frac{v_0 k t^3}{6m} + \frac{16 v_0 \Delta k t^{\frac{7}{2}}}{105 m b^{\frac{1}{2}} \sqrt{\pi}} + \mathcal{O}(t^4) \tag{46}$$

Two different loading cases are studied, and numerical values used in the one-degree-of-freedom examples are $m = 1$ kg, $k = 2$ N/m, and $\Delta k = 1$ N/m, whereas $b = 1$ s in Fig. 1 and $b = 0.02$ s in Fig. 2. The nearly explicit Newmark method for numerical integration, i.e., $\beta = 0$ and $\gamma = \frac{1}{2}$ in Eqs. (38a) and (38b), is employed in both cases. Note that no algorithmic damping is introduced. The time steps chosen are sufficiently smaller than the critical time step in the elastic case. In Fig. 2, we use $\Delta t = 0.1\Delta t_{\text{crit}}^{\text{elastic}}$ based on accuracy considerations. To show the accuracy of the numerical procedure, the numerically obtained displacement is displayed and compared to the time series expansion of the analytical displacement solution in Eq. (46), for a unit pulse loaded one-degree-of-freedom system with $\alpha = \frac{1}{2}$, in Fig. 1. As seen in Fig. 1, the agreement between the numerical solution and the time series expansion is good, which confirms the capability of the numerical procedure. Figure 2

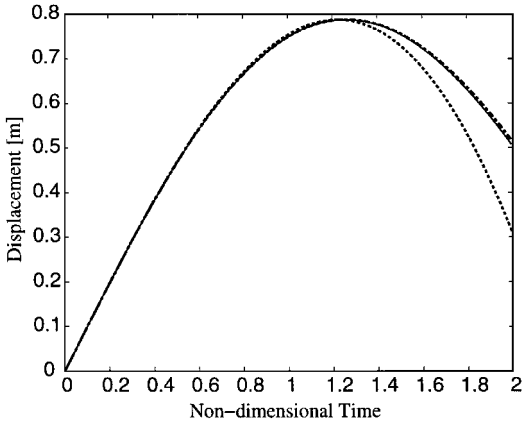

Fig. 1 Displacement vs nondimensional time t/b for a fractionally damped one-degree-of-freedom system subjected to a unit impulsive load at time $t = 0$: ——, numerical solution; .....and - - -, time series expansion of analytically obtained displacement, using three and nine terms, respectively, in the series.

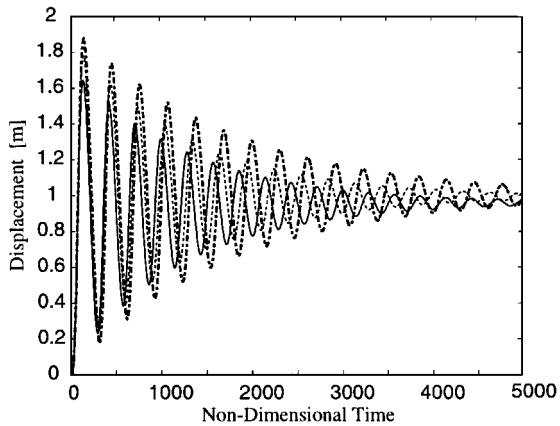

Fig. 2 Displacement vs nondimensional time for a fractionally damped one-degree-of-freedom system subjected to a unit impulse load at time $t = 0$: ——, $\alpha = \frac{1}{3}$; - - -, $\alpha = \frac{1}{2}$; and .....$\alpha = \frac{2}{3}$.

shows numerical displacement solutions in the case of a unit step load. The influence of the fractional derivative exponent α ($\alpha = \frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$) on the displacement is displayed. As seen in Fig. 2, oscillations are more damped with increasing order of the fractional derivative exponent.

B. Viscoelastic Bar

The tip displacement for a transient loaded viscoelastic fixed–free bar is calculated by the present numerical procedure. Moreover, the stability of the solution is exemplified and discussed. The bar is modeled by five linear two-node bar elements. Geometry data for the bar are length = 0.5 m and cross-sectional area = 0.0025 m^2.

To begin, specialize the constitutive equation in Eq. (20) to uniaxial stress states by letting $\sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0$. The corresponding one-dimensional relation to Eq. (20) can then be written as

$$\sigma(t) = E\varepsilon(t) - \Delta E(f * \varepsilon)(t) \tag{47}$$

where $f(t)$ is the kernel in Eq. (22). The bar material has uniform properties, and the following viscous material data for a fictitious polymer are used: $\rho = 1000$ kg/(m^3), $E = 10 \times 10^6$ N/(m^2), $\Delta E = 3 \times 10^6$ N/(m^2), $b = 0.02$ s, and $\alpha = \frac{1}{2}$.

Figure 3 shows the calculated material loss factors, i.e., the ratio between the imaginary and the real parts of the frequency-domain modulus, vs angular frequency for the material under consideration and for a standard viscoelastic material ($\alpha = 1$). As seen in Fig. 3, the frequency dependence of the material loss factor is weaker when using the fractional derivative constitutive law.

The finite bar element used is derived by using a linear displacement interpolation function. For a bar element with a constant cross-

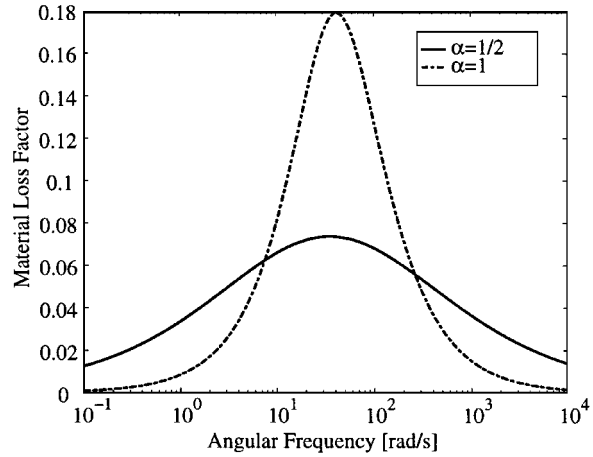

Fig. 3 Material loss factors vs angular frequency for fictitious viscoelastic bar material and standard viscoelastic material, $\alpha = 1$.

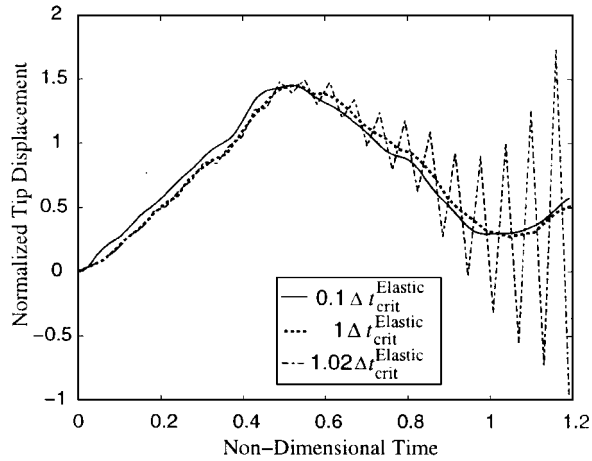

Fig. 4 Normalized tip displacement d/d_{stat} vs nondimensional time t/b for a step loaded viscoelastic bar; coefficient of fractional differentiation $\alpha = \frac{1}{2}$ is used: influence of different time steps is shown, and $\Delta t_{\text{crit}}^{\text{elastic}}$ is the critical time step in the elastic case.

sectional area A^e and length L^e, we have (employing a consistent mass matrix)

$$\boldsymbol{K}^e = \begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix}, \qquad \Delta\boldsymbol{K}^e = \begin{bmatrix} \Delta k^e & -\Delta k^e \\ -\Delta k^e & \Delta k^e \end{bmatrix}$$

$$\boldsymbol{M}^e = \begin{bmatrix} 2m^e & m^e \\ m^e & 2m^e \end{bmatrix}$$

where $k^e = EA^e/L^e$, $\Delta k^e = \Delta E A^e/L^e$, and $m^e = \rho A^e L^e/6$. The lowest and highest elastic or undamped eigenfrequencies of the discretized elastic bar are found to be $\omega_{\min} = 315$ rad/s and $\omega_{\max} = 3333$ rad/s, respectively.

The numerical procedure using the explicit Newmark integration method described in a preceding section is used for the time integration of the nodal displacements. The critical time step for a similar elastic or undamped bar according to Eq. (41) is found to be $\Delta t_{\text{crit}}^{\text{elastic}} = 0.6 \times 10^{-3}$ s. The free end of the bar is excited by a concentrated step load at time $t = 0$, described by a unit step function. The calculated tip displacement is normalized with the quasistatic long time displacement, i.e.,

$$d_{\text{stat}} = \frac{FL}{A(E - \Delta E)} = 0.29 \times 10^{-6}\ \text{m} \tag{48}$$

Tip displacements for the viscoelastic bar using time steps close to the critical time step in the elastic case [see Eq. (41)] are shown in Fig. 4. If the time step chosen is equal to the critical time step, we obtain a stable solution (also for considerably larger times than showed in Fig. 4), whereas a larger time step leads to instability.

This somewhat verifies that taking the critical time step for the elastic case also is convenient for the viscoelastic case.

Figure 5 displays the tip displacement vs time for the bar and loading under consideration, and the time step chosen is $\Delta t = 0.5\,\Delta t_{\text{crit}}^{\text{elastic}}$. Note that the asymptotic value of the displacement for long times slightly differs from the idealized quasistatic long-time value. The reason for this is the time needed for the material to reach the long-time asymptotic value of its stiffness ($E_\infty = E - \Delta E$) is much longer than the time displayed in Fig. 5. Relaxation curves, i.e., the constitutive stress response on a unit strain, presented by Enelund and Olsson,[6] indicate that the time required for the present material to reach a value of its stiffness sufficiently close to its long-time value is in the magnitude of $t/b = 1000$. The results in Fig. 5 have been obtained without truncating the time series for the convolution term in Eq. (37). However, it was found that practically the same result can be obtained by using convolution results from only a few previous time steps.

Figure 6 shows the tip displacement vs time for the bar and, for comparison, the tip displacement for a similar bar of a standard viscoelastic material (obtained by letting the order of differentiations equal one in the constitutive relation). Included in Fig. 6 is a reference solution valid for $\alpha = 1$ obtained using the general purpose FE-code ABAQUS/Explicit.[20] The small deviations observed in Fig. 6, in the high-frequency contributions, can be explained by the different mass representations used. In our approach, a consistent mass matrix is employed [see Eq. (26)], whereas ABAQUS/Explicit uses a lumped mass matrix, i.e., Eq. (26) replaced by a diagonal matrix. Moreover, the explicit time integration routines used in the two codes differ slightly. These deviations diminish if we use a lumped mass matrix approach. However, we prefer to use a consistent mass matrix as it leads to a more accurate solution. Note that α identical to 1 cannot be used in the present algorithm; instead we use $\alpha = 0.999$. As seen in Fig. 6, the fractional calculus model leads to higher damping (in particular, high-frequency contributions are damped out) than the standard model. This is because of the weaker frequency dependence of the material loss factor and because the material loss factor is higher in the frequency range over the lowest elastic eigenfrequency (cf. Fig. 3).

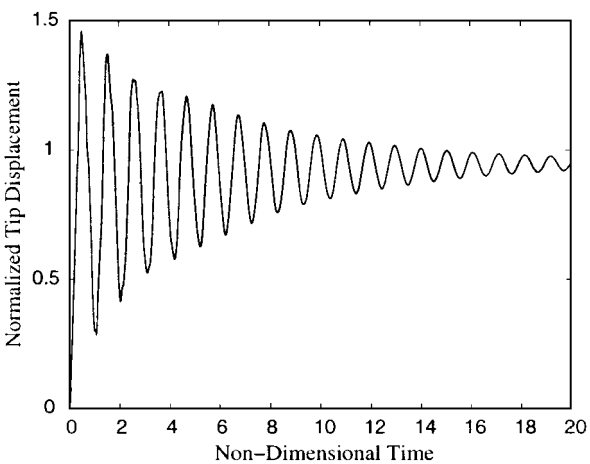

Fig. 5 Normalized tip displacement d/d_{stat} vs nondimensional time t/b for a step loaded viscoelastic bar; coefficient of fractional differentiation equaling $\frac{1}{2}$ is used.

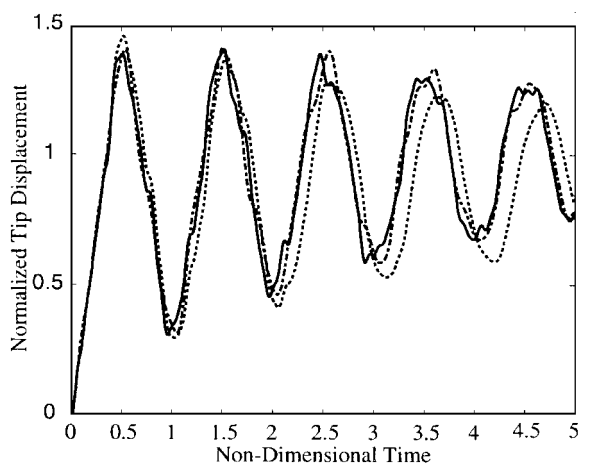

Fig. 6 Normalized tip displacement d/d_{stat} vs nondimensional time t/b for step loaded viscoelastic bars: ——, tip displacement for a bar of standard viscoelastic material, i.e., $\alpha = 1$; ....., tip displacement for a bar of fractional calculus viscoelastic material with $\alpha = \frac{1}{2}$; and - - -, reference solution for the case of $\alpha = 1$.

For larger structures, the time series in Eq. (37) might need to be reduced due to storage requirements and the need to simplify the calculations. This can, of course, be done in several ways, e.g., by simply comparing the contribution from a subsequent term to the sum, bearing in mind that the model represents a fading memory, which as already indicated seems to result in a dependence on a short part of the prehistory.

VI. Summary and Discussion

A three-dimensional isotropic generalization of the fractional calculus model of viscoelasticity using a convolution integral description with a singular kernel function of Mittag–Leffler function type is presented. This formulation is used in the development of a spatially discretized finite element formulation of a viscoelastic structure. The resulting discretized system of equations is a system of integro-differential equations of order two in time. The system constitutes a well-posed initial value problem, which is the main advantage of using the convolution integral formulation. Another advantage of the present formulation is that structures that are viscoelastic only in some parts can be easily handled.

A procedure for a time integration of the resulting discretized system of integro-differential equations is presented. Newmark's method is used for time integration of the nodal degrees of freedom. Special consideration for the time discrete evolution of convolution term is required. A special analytical property of the kernel function, together with Grünwald's definition of fractional differentiation, and a backward Euler rule are used, which enables us to calculate the convolution term. This requires that the actual nodal degrees of freedom and the complete previous history of the convolutions are known. A very desirable feature is that no actual evaluation of the kernel function is needed. In the case of the nearly explicit Newmark method, the procedure becomes very simple and the unknown nodal degrees of freedom may be calculated directly.

A few numerical examples for transient loaded one-degree-of-freedom systems and bar systems are presented. In one example, we compare the numerically obtained solution with an established time series expansion of an analytical solution. The agreement is very good. Moreover, the calculated displacements indicate that the present numerical procedure is accurate and potentially very useful for time integration of viscoelastic systems with a very flexible description of the inelasticity and the damping.

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